AWS 2008 Project updated: *Aug 22, 2008*

Algebraic Values of Transcendental Functions at Algebraic Points

by

Jingjing Huang, Diego Marques and Martin Mereb

Abstract

In this paper, the authors will prove that any subset of $\overline{\mathbb{Q}}$ can be the exceptional set of some transcendental entire function. Furthermore, we could generalize this theorem to a much more general version and present a unified proof.

1 Introduction and the main result

In 1886, Weierstrass gave an example of a transcendental entire function which takes rational values at all rational points. He also suggested that there exist transcendental entire functions which take algebraic values at any algebraic point. Later, in [3], Stäckel proved that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there is a transcendental entire function f such that $f(\Sigma) \subseteq T$. Another construction due to Stäckel produces an entire function f whose derivatives $f^{(s)}$, for $s = 0, 1, 2, \dots$, all map $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}$, see [4]. A more thorough discussion on this subject can be found in [1]. There are recent results due to Andrea Surroca on the number of algebraic points where a transcendental analytic function takes algebraic values, see [2]. We were able to generalize these two results of Stäckel to the following general theorem.

Theorem 1. *Given a countable subset $A \subseteq \mathbb{C}$ and for each integer $s \geq 0$ with $\alpha \in A$, fix a dense subset $E_{\alpha,s} \subseteq \mathbb{C}$. Then there exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f^{(s)}(\alpha) \in E_{\alpha,s}$, for all $\alpha \in A$ and all $s \geq 0$.*

Let f be given, and denote by S_f the set of all algebraic points $\alpha \in \mathbb{C}$, for which $f(\alpha)$ is also algebraic. An interesting problem is to determine properties of S_f, which we name as the exceptional set of f. In the conclusion we will show that for any $A \subseteq \overline{\mathbb{Q}}$ there is a transcendental entire function f such that A is the exceptional set of f.

Without referring to Theorem 1, we have the following special examples:

Example 1. *Arbitrary finite subsets of algebraic numbers are easily seen to be exceptional. For instance, if* $f_1(z) = e^{(z-\alpha_1)\cdots(z-\alpha_k)}$, *then the Hermite-Lindemann theorem implies* $S_{f_1} = \{\alpha_1, \ldots, \alpha_k\}$. *If* $f_2(z) = e^z + e^{z+1}$ *and* $f_3(z) = e^{z\pi+1}$, *then the Lindemann-Weierstrass and Baker theorems imply* $S_{f_2} = S_{f_3} = \varnothing$.

Example 2. *Some well-known infinite sets are also exceptional, for instance, if* $f_4(z) = 2^z, f_5(z) = e^{i\pi z}$, *then* $S_{f_4} = S_{f_5} = \mathbb{Q}$, *by the Gelfond-Schneider theorem.*

Example 3. *Assuming Schanuel's conjecture to be true, it is easy to prove that if* $f_6(z) = \sin(\pi z)e^z, f_7(z) = 2^{3^z}$ *and* $f_8(z) = 2^{2^{2^{z-1}}}$, *then* $S_{f_6} = S_{f_7} = \mathbb{Z}$ *and* $S_{f_8} = \mathbb{N}$.

These examples are just special case of our Theorem 1, hitherto can be proved uniformly here.

2 Preliminary results

Before going to the proof of the theorem, we need couple of lemmas.

Lemma 1. *Let* $\{P_n(z)\}_{n\geq 0}$ *be a sequence of complex polynomials, where* $\deg P_n = n$. *And let* $\{C_n\}_{n\geq 0}$ *be a sequence of positive constants providing that* $|P_n(z)| \leq C_n \max\{|z|, 1\}^n$. *If a sequence of complex numbers* $\{a_n\}_{n\geq 0}$ *satisfies* $|a_n| \leq \frac{1}{C_n n!}$, *then the series* $\sum_{n=0}^{\infty} a_n P_n(z)$ *converges absolutely and uniformly on any compact sets, particularly this gives an entire function.*

Proof. When $|a_n| \leq \frac{1}{C_n n!}$, we have:

$$\sum_{n=0}^{\infty} |a_n||P_n(z)| \leq \sum_{n=0}^{\infty} \frac{1}{C_n n!} C_n \max\{|z|, 1\}^n \leq \exp(\max\{|z|, 1\}),$$

so $\sum_{n=0}^{\infty} a_n P_n(z)$ converges absolutely and uniformly on any compact sets. Therefore this series will produce an entire function. $\square$

Now, let's enumerate the set A in Theorem 1 as $\{\alpha_1, \alpha_2, \alpha_3, \cdots\}$

For $n \geq 1$, setting $n = 1 + 2 + 3 + \cdots + m_n + j_n$, where $m_n \geq 0$ and $1 \leq j_n \leq m_n + 1$. Next, construct a sequence of polynomials as follows

$$P_0(z) = 1 \text{ and define recursively } P_n(z) = (z - \alpha_{j_n}) P_{n-1}(z) \text{ for } n \geq 1$$

Here we list the first few polynomials:

$$\begin{aligned}
P_0(z) &= 1 \\
P_1(z) &= (z - \alpha_1) \\
P_2(z) &= (z - \alpha_1)^2 \\
P_3(z) &= (z - \alpha_1)^2 (z - \alpha_2) \\
P_4(z) &= (z - \alpha_1)^3 (z - \alpha_2) \\
P_5(z) &= (z - \alpha_1)^3 (z - \alpha_2)^2 \\
P_6(z) &= (z - \alpha_1)^3 (z - \alpha_2)^2 (z - \alpha_3) \\
P_7(z) &= (z - \alpha_1)^4 (z - \alpha_2)^2 (z - \alpha_3) \\
&\vdots
\end{aligned}$$

For convenience, let's denote $i_n = m_n + 1 - j_n$. For any given $i \geq 0$ and $j \geq 1$ there exists a unique $n \geq 1$ such that $i_n = i$ and $j_n = j$, namely $n = \frac{(i+j)(i+j-1)}{2} + j$.

Lemma 2. *For $n \geq 1$, we have $P_{n-1}^{(i_n)}(\alpha_{j_n}) \neq 0$ and $P_l^{(i_n)}(\alpha_{j_n}) = 0$ when $l \geq n$.*

Proof. From the definition of $P_n(z)$, we can write explicitly

$$P_l(z) = (z - \alpha_1)^{m_l} (z - \alpha_2)^{m_l - 1} \cdots (z - \alpha_{m_l})(z - \alpha_1) \cdots (z - \alpha_{j_l})$$

It follows that α_{j_n} is a zero of $P_{n-1}(z)$ with multiplicity i_n, which means $P_{n-1}^{(i_n)}(\alpha_{j_n}) \neq 0$. On the other hand, if $l \geq n$, then α_{j_n} is a zero of $P_l(z)$ with multiplicity at least $i_n + 1$, which implies $P_l^{(i_n)}(\alpha_{j_n}) = 0$. $\square$

Lemma 3. *If $\sum_{k=0}^{\infty} a_k P_k(z) = \sum_{k=0}^{\infty} b_k P_k(z)$ for all $z \in \mathbb{C}$, then $a_k = b_k$ for each $k \geq 0$.*

Proof. It suffice to prove that if $g(z) := \sum_{k=0}^{\infty} a_k P_k(z) = 0$ for all $z \in \mathbb{C}$, then $\{a_k\}_{k\geq 0}$ is identically 0. Notice that $a_0 = g(\alpha_1) = 0$. Assuming $a_0, a_1, \dots, a_{n-1}$ are all 0, by Lemma 2, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) \\ &= \sum_{k=0}^{n-1} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) + a_n P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) + \sum_{k=n+1}^{\infty} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) \\ &= a_n P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \end{aligned}$$

Since $P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \neq 0$, we have $a_n = 0$. Hence the proof will be completed by induction.

□

Now we are able to prove our theorem.

3 Proof of the theorem

We are going to construct the desired transcendental entire function by fixing the coefficients in the series $\sum_{k=0}^{\infty} a_k P_k(z)$ recursively, where the sequence $\{P_k\}_{k\geq 0}$ has been defined in Section 2.

First, by Lemma 1, the condition $|a_k| \leq \frac{1}{C_k k!}$ will ensure $\sum_{k=0}^{\infty} a_k P_k(z)$ to be entire.

Now we will fix the coefficients a_k recursively. For $n \geq 1$, we denote $E_n = E_{\alpha_{j_n}, i_n}$ and let the numbers $\beta_n = \sum_{k=0}^{\infty} a_k P_k^{(i_n)}(\alpha_{j_n})$. We are going to choose the value of a_k so that $\beta_n \in E_{\alpha_{j_n}, i_n} = E_n$ for all $n \geq 1$.

By Lemma 2, we know that $P_l^{(i_n)}(\alpha_{j_n}) = 0$ when $l \geq n$, so β_n is actually the finite sum $\sum_{k=0}^{n-1} a_k P_k^{(i_n)}(\alpha_{j_n})$. Notice that $\beta_1 = a_0 P_0^{(0)}(\alpha_1) = a_0$ and E_1 is dense, we can fix a value for a_0 such that $0 < |a_0| \leq \frac{1}{C_0}$ and $\beta_1 \in E_1$. Now suppose that the values of $\{a_0, a_1, \cdots, a_{n-1}\}$ are well fixed such that $0 < |a_k| \leq \frac{1}{C_k k!}$ and $\beta_k \in E_k$ for $0 \leq k \leq n-1$. By Lemma 2, we know $P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \neq 0$, so we can pick a proper value of a_n such that $0 < |a_n| \leq \frac{1}{C_n n!}$ and $\beta_n = \sum_{k=0}^{n-1} a_k P_k^{(i_{n+1})}(\alpha_{j_{n+1}}) + a_n P_n^{(i_{n+1})}(\alpha_{j_{n+1}}) \in E_n$.

So now by induction all the a_k are well chosen such that for all $k \geq 1$ we have $0 < |a_k| \leq \frac{1}{C_k k!}$ and $\beta_k \in E_k$. Thus by Lemma 1, the function $f(z) = \sum_{k=0}^{\infty} a_k P_k(z)$ is an entire function and for any

$i \geq 0, j \geq 1$ we have $f^{(i)}(\alpha_j) = \sum_{k=0}^{\infty} a_k P_k^{(i)}(\alpha_j) = \beta_n \in E_n = E_{\alpha_j,i}$ where n is the unique integer such that $i_n = i, j_n = j$. Taking into account that every polynomial could be expressed as a finite linear combination of the $\{P_k\}$, and all the $\{a_k\}$ here are not 0, so by Lemma 3 we conclude that $f(z)$ is not a polynomial. Hence $f(z)$ is the desired transcendental entire function.

From the construction of the proof, we can easily see that in fact there are uncountably many functions satisfying the properties required in Theorem 1.

4 Applications to Exceptional Sets

We recall the following definition

Definition 1. *Let f be an entire function. We define* the exceptional set of f *to be*

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}}\}$$

We list some of the more interesting consequences of Theorem 1 with the choice of $A,\ E_{\alpha,s}$ noted in parentheses.

Corollary 1. *For each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$ there is a transcendental entire function f such that $f^{(s)}(\Sigma) \subseteq T$ for $s \geq 0$. ($A = \Sigma,\ E_{\alpha,s} = T$)*

Corollary 2. *Let $A \subseteq \mathbb{C}$ be countable and dense in $\mathbb{C}$, then there is a transcendental entire function f such that $f^{(s)}(A) \subseteq A$, for $s \geq 0$. ($E_{\alpha,s} = A$)*

Corollary 3. *There exists a transcendental entire function such that $f^{(s)}(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}(i)$, for $s \geq 0$. ($A = \overline{\mathbb{Q}},\ E_{\alpha,s} = \mathbb{Q}(i)$)*

The next result shows that in particular every $A \subseteq \overline{\mathbb{Q}}$ is an exceptional set of a transcendental entire function.

Theorem 2. *If $A \subseteq \overline{\mathbb{Q}}$, then there is a transcendental entire function, such that $S_{f^{(s)}} = A$ for $s \geq 0$.*

Proof. Suppose that A and $\overline{\mathbb{Q}} \backslash A$ are both infinite, thus we can enumerate $\overline{\mathbb{Q}} = \{\alpha_1, \alpha_2, \ldots\}$ where $A = \{\alpha_1, \alpha_3, ..., \alpha_{2n+1}, ...\}$. Set $E_{\alpha_{2n+2},s} = \mathbb{C} \backslash \overline{\mathbb{Q}}$ and $E_{\alpha_{2n+1},s} = \overline{\mathbb{Q}}$ for all $n, s \geq 0$. Now by Theorem 1, there exists a transcendental entire function f with $f^{(s)}(\alpha_{2n+1}) \in \overline{\mathbb{Q}}$ and

$f^{(s)}(\alpha_{2n+2}) \in \mathbb{C}\backslash\overline{\mathbb{Q}}$, for each $n, s \geq 0$. Therefore it is plain that $S_{f^{(s)}} = A$.

For the case that A is finite, we can suppose $A = \{\alpha_1, ..., \alpha_m\}$. Take $E_{\alpha_1,s} = \cdots = E_{\alpha_m,s} = \overline{\mathbb{Q}}$ for all $s \geq 0$, and set $E_{\alpha_k,s} = \mathbb{C}\backslash\overline{\mathbb{Q}}$ for all $k > m, s \geq 0$. In case, $\overline{\mathbb{Q}}\backslash A = \{\alpha_1, ..., \alpha_m\}$, we take $E_{\alpha_1,s} = \cdots E_{\alpha_m,s} = \mathbb{C}\backslash\overline{\mathbb{Q}}$ for all $s \geq 0$, and set $E_{\alpha_k,s} = \overline{\mathbb{Q}}$ for all $k > m, s \geq 0$. Then for these two cases we proceed as in the proof above.

□

Acknowledgements

The authors would like to express their gratitude to Professor Michel Waldschmidt for his instruction and guidance. We also thank Georges Racinet for providing encouragement and assistance during the evening workshops. Additionally, a debt is owed to all of the students who participated; Chuangxun Cheng, Brian Dietel, Mathilde Herblot, Holly Krieger, Jonathan Mason and Robert Wilson. Finally, we must thank the organizers of the 2008 Arizona Winter School and the hospitality of the University of Arizona for making this work possible. The second author is a scholarship holder of CNPq. The third is supported by Harrington fellowship.

References

[1] K. Mahler, *Lectures on transcendental numbers,* Lectures notes in mathematics, Vol. 546. Springer-Verlag, Berlin-New York, 1976.

[2] A. Surroca, *Valeurs algébriques de fonctions transcendantes,* Int. Math. Res. Not. (2006), Art. ID 16834, 31 pages.

[3] P. Stäckel, *Ueber arithmetische Eingenschaften analytischer Functionen,* Mathematische Annalen 46 (1895), no. 4, 513–520.

[4] P. Stäckel, *Arithmetische Eingenschaften analytischer Functionen,* Acta Mathematica 25 (1902), 371–383.

Jingjing Huang (Penn State U.) huang@math.psu.edu
Diego Marques (U. Federal do Ceara) diego@mat.ufc.br
Martin Mereb (U. Texas) mmereb@gmail.com